

A variational approach to the QCD wavefunctional:
Calculation of the QCD β -function.

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ABSTRACT

The β -function is calculated for an $SU(N)$ Yang-Mills theory from an ansatz for the vacuum wavefunctional. Direct comparison is made with the results of calculations of the β -function of QCD. In both cases the theories are asymptotically free. The only difference being in the numerical coefficient of the β -function, which is found to be -4 from the ansatz and $-4 + \frac{1}{3}$ from other QCD calculations. This is because, due to the constraint of Gauss' law applied to the wavefunctional, transverse gluons (which contribute the $\frac{1}{3}$) are omitted. The renormalisation procedure is understood in terms of 'tadpole' and 'horse-shoe' Feynman diagrams which must be interpreted with a non-local propagator.

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1. Introduction.

One of the main problems in modern quantum field theory is the understanding of low energy phenomena in QCD, such as confinement, chiral symmetry breaking, or, in more general terms, the strong coupling problem. To have analytic results for the ground state of an asymptotically free non-abelian gauge theory, with the associated enhanced understanding of the underlying physics, would be invaluable in the understanding of these phenomena. Although many promising ideas have been suggested in the first nearly quarter century of QCD, e.g. [1], we are still far from a completely satisfactory answer.

The arsenal of non-perturbative methods available to tackle strongly interacting continuum theories is limited. Although in simple quantum mechanical problems a variational approach is often easy to use - it is usually enough to know a few simple qualitative features in order to set up a variational ansatz that will give good results for the ground state energy and other vacuum expectation values - this is not the case in quantum field theories, the complexities of which pose difficult problems, as discussed by Feynman, [2].

Recently, a variational approach based on a gauge invariant Gaussian wavefunctional has been studied. This method was applied to QCD and QED₃, [3] and [4] respectively, where, although in its infancy, it has independently verified many old, and given some new, results. A Gaussian approach to the wavefunctional of QCD (the so-called squeezed gluons) has been studied in many different papers, see [5] and the references therein. In the case of QED₃ the variational method reproduced Polyakov's path integral results of the mass gap and string tension of the theory, [4].

The variational calculation carried out for the SU(N) purely Yang-Mills theory in 3+1 dimensions, [3], has found that the ground state energy is minimal for a state which is different from the perturbative vacuum even though the perturbative vacuum state was

included in the variational ansatz. Dynamical scale generation takes place and the gluon (SVZ) condensate in the best variational state was found to be non-zero.

The ansatz used for the vacuum wavefunctional was Gaussian but it was also required to be gauge invariant. To satisfy this, the Gaussian wavefunctional is projected onto the gauge invariant sector. The form of the wavefunctional is discussed briefly in the next section and in detail in [3]. The variational parameter of the theory is the mass scale, M . A non-zero value of M for the minimal state corresponds to a non-perturbative dynamical scale generation.

Within the variational calculation of [3] it was conjectured that the variational ansatz proposed yields a theory with a coupling constant that runs as the coupling constant of (asymptotically free) QCD. In this paper we prove this conjecture. It is shown that the proposed variational ansatz yields an effective non-local, non-linear sigma model in three dimensions which, when renormalised to first order, has practically (a precise qualification of this word will be given below) the same β -function as asymptotically free QCD. Further, the renormalisation procedure is interpreted in terms of the Feynman diagrams included and it is found that, due to the non-local nature of the propagator, a new ‘horse-shoe’ diagram is non-zero and makes a vital contribution.

In this paper, the β -function for the charge of the variational ansatz is calculated to be

$$\beta(g) = -\frac{g^3}{(4\pi)^2}4C_2(G) + O(g^5) \quad (1.1)$$

This should be directly compared with the known β function for QCD, [9];

$$\beta(g) = -\frac{g^3}{(4\pi)^2}[(4 - \frac{1}{3})C_2(G) - \frac{2}{3}n_f C(r)] + O(g^5) \quad (1.2)$$

n_f is the number of species of fermions in representation r , which is zero for the purely Yang-Mills model. For $SU(N)$, $C_2(G)$, the quadratic Casimir in the adjoint representation,

is N . We immediately see that the only difference between the two β -functions is in the numerical terms 4 and $(4 - \frac{1}{3})$. The 4 is due to the anti-screening effect of longitudinal gluons. The $\frac{1}{3}$ is due to the screening effect of virtual transverse gluons. Identically the same β -function as QCD is not obtained because only longitudinal gluons have been included within the ansatz. The longitudinal gluons are included because they satisfy Gauss' law with a source, which is used as a constraint upon the vacuum wavefunctional in setting up the variational ansatz.

In this paper the renormalisation calculation to first order is presented and the renormalisation of the effective charge in the variational wavefunctional is obtained. In the first section we shall describe the variational ansatz of [3] and show how this leads to an effective non-local, non-linear sigma model in three dimensions. In the second section we perform the renormalisation group transformation, integrating over high momentum dependent modes to yield an effective action for the low momentum modes with a renormalised coupling constant, and we interpret the renormalisation procedure by considering the Feynman diagrams which contribute.

2. The variational ansatz.

For a full discussion of the variational ansatz and all details of the subsequent variational calculation the reader is directed to the original paper, [3]. An overview of the variational ansatz and the form of the effective action is given in this section.

The $SU(N)$ gauge theory is described by the Hamiltonian,

$$H = \int d^3x \left[\frac{1}{2} E_i^{a2} + \frac{1}{2} B_i^{a2} \right] \quad (2.1)$$

where

$$\begin{aligned} E_i^a(x) &= i \frac{\delta}{\delta A_i^a(x)} \\ B_i^a(x) &= \frac{1}{2} \epsilon_{ijk} \{ \partial_j A_k^a(x) - \partial_k A_j^a(x) + g f^{abc} A_j^b(x) A_k^c(x) \} \end{aligned} \quad (2.2)$$

and all physical states must satisfy the constraint of gauge invariance (Gauss' law);

$$G^a(x) \Psi[A] = \left[\partial_i E_i^a(x) - g f^{abc} A_i^b(x) E_i^c(x) \right] \Psi[A] = 0 \quad (2.3)$$

Under a gauge transformation U (generated by $G^a(x)$) the vector potential transforms as

$$A_i^a(x) \rightarrow A_i^{Ua}(x) = S^{ab}(x) A_i^b(x) + \lambda_i^a(x) \quad (2.4)$$

where

$$S^{ab}(x) = \frac{1}{2} \text{tr} \left(\tau^a U^\dagger \tau^b U \right); \quad \lambda_i^a(x) = \frac{i}{g} \text{tr} \left(\tau^a U^\dagger \partial_i U \right) \quad (2.5)$$

and τ^a are traceless Hermitian N by N matrices satisfying $\text{tr}(\tau^a \tau^b) = 2\delta^{ab}$. For $SU(3)$ the algebra and structure constants are defined, for example, in [8].

The initial ansatz for the ground state wavefunctional is of vital importance. It must incorporate the properties of all such physical states and yet it must not lead to a solution which is incalculable if any progress is to be made. In this formalism one calculates expectation values of local operators with the ansatz for the ground state, Ψ ,

$$\langle O \rangle = \frac{1}{Z} \int D\phi \Psi^*[\phi] O \Psi[\phi] \quad (2.6)$$

and then minimises with respect to the variational parameter. A calculation of this kind is tantamount to evaluation of a Euclidean path integral with the square of the wavefunctional playing the role of the partition function. One should therefore be able to solve exactly a d -dimensional field theory with the action

$$S[\phi] = -\log \Psi^*[\phi] \Psi[\phi] \quad (2.7)$$

Since in dimension $d > 1$ the only theories one can solve exactly are free field theories, the requirement of calculability almost unavoidably restricts the possible form of the wavefunctional to a Gaussian (or, as it is sometimes called, squeezed) state:

$$\Psi[A_i^a] = \exp \left\{ -\frac{1}{2} \int d^3x d^3y [A_i^a(x) - \zeta_i^a(x)] (G^{-1})_{ij}^{ab}(x, y) [A_j^b(y) - \zeta_j^b(y)] \right\} \quad (2.8)$$

with $\zeta(x)$ and $G(x, y)$ being c-number functions. The requirement of translational invariance usually gives further restrictions: $\zeta(x) = \text{const}$, $G(x, y) = G(x - y)$.

There is, however, one obvious difficulty with this idea. It is impossible to write down a Gaussian wavefunctional which satisfies the constraint of gauge invariance. Under the gauge transformation the wavefunctional transforms as

$$\Psi[A_i^a] \rightarrow \Psi[(A^U)_i^a] \quad (2.9)$$

In the abelian case it is enough to take $\partial_i G_{ij}^{-1} = 0$ to satisfy the constraint of gauge invariance. In the non-abelian case, however, due to the homogeneous piece in the gauge transformation (2.4), no gauge invariant Gaussian wavefunctional exists.

The proposed solution to this problem was to simply project the Gaussian wavefunctional onto the gauge invariant sector and to restrict the calculation to the case of zero classical fields ($\zeta = 0$). The variational ansatz is therefore

$$\Psi[A_i^a] = \int DU(x) \exp \left\{ -\frac{1}{2} \int d^3x d^3y A_i^{Ua}(x) G_{ij}^{-1ab}(x - y) A_j^{Ub}(y) \right\} \quad (2.10)$$

with A_i^{Ua} defined in (2.4) and the integration performed over the space of special unitary matrices with the $SU(N)$ group invariant measure.

Further restrictions upon the form of G lead to considerable simplifications. Firstly, only matrices of the form

$$G_{ij}^{ab}(x - y) = \delta^{ab} \delta_{ij} G(x - y) \quad (2.11)$$

are considered. This is certainly the correct form in the perturbative regime. If it was not for the integration over the group, G_{ij}^{-1} would be precisely the (equal time) propagator of the electric field. Due to the integration over the group, however, the actual propagator is the transverse part of G^{-1} . The longitudinal part $\partial_i G_{ij}^{-1}$ drops out of all physical quantities, giving, without any loss of generality at the perturbative level, $G_{ij} \sim \delta_{ij}$. Also, in the leading order in perturbation theory, the non-abelian character of the gauge group is not important. The δ^{ab} structure is then obvious.

The form of G can be restricted further using additional perturbative information. The theory is asymptotically free. This means that the short distance asymptotics of correlation functions must be the same as in the perturbation theory. Since G^{-1} is directly related to correlation functions of gauge invariant quantities in perturbation theory, it is taken to have the form,

$$G^{-1}(x) \rightarrow \frac{1}{x^4}, \quad x \rightarrow 0 \quad (2.12)$$

The non-perturbative theory is also expected to have a gap. In other words, the correlation functions should decay to zero at some distance scale,

$$G(x) \sim 0, \quad x > \frac{1}{M} \quad (2.13)$$

The variational ansatz is built in the simplest possible way. M is taken to be the only variational parameter and this is done by choosing $G(x)$ to be of a form that has the ultra-violet and infra-red asymptotics of (2.12) and (2.13). A non-zero result for M means a non-perturbative dynamical scale generation in the Yang-Mills vacuum. The form of G^{-1} used in this paper and the variational calculation of [3] has the Fourier transform

$$G^{-1}(k) = \begin{cases} \sqrt{k^2} & \text{if } k^2 > M^2 \\ M & \text{if } k^2 < M^2 \end{cases} \quad (2.14)$$

Equation (2.10) together with equations (2.11) and (2.14) define our variational ansatz.

It will now be shown that the action (2.7) is in fact a non-local, non-linear sigma model. Again, the reader is referred to [3] for the original account. With the given ansatz, the remaining problem is to calculate expectation values of local operators, such as,

$$\langle O \rangle = \frac{1}{Z} \int DU DU' \langle O \rangle_A \quad (2.15)$$

where

$$\begin{aligned} \langle O \rangle_A &= \int DA e^{-\frac{1}{2} \int dx dy A_i^{Ua}(x) G^{-1}(x-y) A_i^{Ua}(y)} O e^{-\frac{1}{2} \int dx' dy' A_j^{U'b}(x') G^{-1}(x'-y') A_j^{U'b}(y')} \\ &= \int DA e^{-\frac{1}{2} \int dx dy A_i^{Ua}(x) G^{-1}(x-y) A_i^{Ua}(y)} O e^{-\frac{1}{2} \int dx' dy' A_j^b(x') G^{-1}(x'-y') A_j^b(y')} \end{aligned} \quad (2.16)$$

where, since only gauge invariant operators are to be considered, the change of variable $A_i \rightarrow A_i^{-U'}$ has rendered one of the group integrations redundant.

For convenience, the definition

$$a_i^a(x) = \int d^3 y d^3 z \lambda_i^b(y) G^{-1}(y-z) S^{bc}(z) (\mathcal{M}^{-1})^{ca}(z, x) \quad (2.17)$$

is made so that the Gaussian integration over A_i is $\int DA \exp[-\frac{1}{2}(A+a)\mathcal{M}(A+a)]$. Changing variables and performing the integration yields the following form of the normalization factor Z ,

$$Z = \int DU \exp\{-\Gamma[U]\} \quad (2.18)$$

with an effective action

$$\Gamma[U] = \frac{1}{2} \text{Tr} \ln \mathcal{M} + \frac{1}{2} \lambda^a \Delta^{ac} \lambda^c \quad (2.19)$$

where

$$\Delta^{ac}(x, y) = [G(x-y)\delta^{ac} + S^{ab}(x)G(x-y)S^{Tbc}(y)]^{-1} \quad (2.20)$$

is the ‘effective inverse propagator’ and multiplication is understood as the matrix multiplication with indices: colour a , space i and position (the values of space coordinates) x ,

i.e.

$$(AB)_{ik}^{ac}(x, z) = \int d^3y A_{ij}^{ab}(x, y) B_{jk}^{bc}(y, z), \quad \lambda O \lambda = \int d^3x d^3y \lambda_i^a(x) O_{ab}^{ij}(x - y) \lambda_j^b(y) \quad (2.21)$$

The trace Tr is understood as a trace over all three types of indices. In equation (2.19) we have defined

$$S_{ij}^{ab}(x, y) = S^{ab}(x) \delta_{ij} \delta(x - y), \quad \mathcal{M}_{ij}^{ab}(x, y) = [S^{Tac}(x) S^{cb}(y) + \delta^{ab}] G^{-1}(x - y) \delta_{ij} \quad (2.22)$$

where $S^{ab}(x) = \frac{1}{2} \text{tr} (\tau^a U^\dagger \tau^b U)$ and $\lambda_i^a(x) = \frac{i}{g} \text{tr} (\tau^a U^\dagger \partial_i U)$ were defined in (2.5) and tr is a trace over colour indices only. One should also note here another useful definition, the completeness condition for SU(N);

$$\tau_{ij}^a \tau_{kl}^a = 2(\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl}) \quad (2.23)$$

The path integral (2.18) defines a partition function of a non-linear sigma model in three dimensional Euclidean space. The action of this sigma model is rather complicated. It is a non-local and a non-polynomial functional of $U(x)$. We shall see how the coupling appears in the effective action in both high and low momentum cases. Various approximations are made here which are explored much more rigorously in the calculation of the succeeding section.

For high momentum modes, with the standard parametrization $U(x) = \exp[i\frac{g}{2}\phi^a(x)\tau^a]$, one gets $\lambda_i^a(x) = -\partial_i \phi^a(x) + O(g)$, $S^{ab}(x) = \delta^{ab} + O(g)$ and the leading order term in the action becomes:

$$\frac{1}{4} \int d^3x d^3y \partial_i \phi^a(x) G^{-1}(x - y) \partial_i \phi^a(y) \quad (2.24)$$

This is just a free theory with a non-standard propagator;

$$\langle \phi^a(x) \phi^b(y) \rangle = 2\delta^{ab} [\partial_i^x \partial_i^y G^{-1}(x - y)]^{-1} = 2\delta^{ab} \int \frac{d^3k}{(2\pi)^3} \frac{\exp[ik \cdot (x - y)]}{k^3} \quad (2.25)$$

For the low momentum modes, to a first level of approximation, the space dependence of $S^{ab}(x)$ is ignored in the term SGS^T giving, with the fact that S is an orthogonal matrix, the approximation $SGS^T \rightarrow G$. Then, using the completeness condition, (2.23), and the fact that $\text{tr}(U^\dagger \partial_i U) = 0$ we can write

$$\lambda_i^a(x) \lambda_i^a(x) = -(1/g^2) \text{tr} \left(\tau^a U^\dagger \partial_i U \right) \text{tr} \left(\tau^a U^\dagger \partial_i U \right) = -(2/g^2) \text{tr} \left(U^\dagger \partial_i U U^\dagger \partial_i U \right) \quad (2.26)$$

In this approximation the action becomes

$$\frac{1}{2g^2} \text{tr} \int d^3x d^3y \partial_i U^\dagger(x) G^{-1}(x-y) \partial_i U(y) = \frac{M}{2g^2} \text{tr} \int d^3x \partial_i U^\dagger(x) \partial_i U(x) + \dots \quad (2.27)$$

where $+\dots$ corresponds to all the higher terms in g .

3. Calculation of the β -function.

Having obtained the effective action of (2.19), which is a non-local, non-linear sigma model in three dimensions, we shall proceed to calculate the β function. The form of the β -function is deduced from the renormalised coupling constant, in the spirit of [7] where a similar calculation was performed for the non-linear sigma model in two dimensions. We shall perform the renormalisation group transformation by integrating over high momentum dependent modes (containing Fourier components $k > M$) leaving an effective action for low momentum dependent modes (containing Fourier components $k < M$). The coupling constant of the effective action is renormalised to first order; up to terms quadratic in the high momentum modes. A physical interpretation of these terms by considering the corresponding Feynman diagrams is given in section 4. The high momentum modes are the quantum field, and the low momentum modes are the classical field, of the background field method.

Next we shall discuss the decomposition of the group elements into high and low momentum dependent modes and in the second subsection we shall explicitly calculate the β -function.

3.1 Quadratic approximation for the high momentum modes.

The ansatz proposed for the decomposition of group elements into high and low momentum dependent modes is

$$U(x) = U_L(x)U_H(x) \quad (3.1)$$

where $U_L(x)$ contains Fourier components $k < M$ and $U_H(x)$ contains Fourier components $k > M$. This can be considered as a decomposition of the group parameter, $\phi^a(x)$. If we write $U(x) = \exp[\frac{ig}{2}\phi^a(x)\tau^a]$ then the decomposition can be written as $\phi^a(x) = \phi_H^a(x) + \phi_L^a(x)$, where $\phi_{H,L}^a(x)$ are the group parameters of $U_{H,L}(x)$, respectively, and ϕ_H^a and ϕ_L^a are taken to be orthogonal; $\phi_H^a(x)\phi_L^a(x) = 0$.

Written in terms of the group parameters, we can explicitly see that this ansatz has similarities to that used by Polyakov in his treatment of the non-linear sigma model in two dimensions, [7], but without the normalisation of $\phi_L^a(x)$ constrained to be that of $\phi^a(x)$. In his ansatz, Polyakov forced $|\phi^a|^2 = |\phi_L^a|^2$, proposing

$$\phi^a(x) = \phi_L^a(x)(1 - |\phi_H^a|^2)^{\frac{1}{2}} + \phi_{i,H}(x)e_i^a(x) \quad (3.2)$$

which has been written in the notation of this paper. $e_i^a(x)$ form a complete basis of unit vectors orthonormal to $\phi_L^a(x)$. We do not use this ansatz for the decomposition, however, because for each order of g in the calculation it mixes the high and low momentum group parameters making the desired decomposition (3.1) difficult.

For the rest of this section we shall employ the decomposition (3.1) to write the

effective Lagrangian for $\lambda_{i,L}^a$ up to terms of $O(g^2)$. This corresponds to all terms up to, and including, those quadratic in the field ϕ .

It is necessary to first note two identities for λ_i^a and S^{ab} . Using the completeness condition for $SU(N)$, (2.23), with the cyclic property of the trace and the fact that $\text{tr}[U\tau^a U^\dagger] = \text{tr}[\tau^a] = 0$ we find,

$$S^{ac}(x) = S_H^{ab}(x)S_L^{bc}(x) \quad (3.3)$$

and

$$\lambda_i^a(x) = S_H^{ab}\lambda_{i,L}^b(x) + \lambda_{i,H}^a(x) \quad (3.4)$$

Using the same mathematical properties, we should also note that S^{ab} is an orthogonal matrix,

$$S^{ab}(x)S^{Tbc}(x) = S^{ab}(x)S^{cb}(x) = \frac{1}{4}\tau_{ij}^b\tau_{kl}^b(U(x)\tau^a U^\dagger(x))_{ij}(U(x)\tau^c U^\dagger(x))_{kl} = \frac{1}{2}\text{tr}[\tau^a\tau^c] = \delta^{ac} \quad (3.5)$$

First, we shall evaluate the inverse effective propagator, $\Delta^{ac}(x, y)$ (2.20), up to terms quadratic in the coupling constant. To do this we write,

$$U_H(x) = \exp\left[\frac{ig}{2}\phi^a(x)\tau^a\right] = 1 + \frac{ig}{2}\phi^a(x)\tau^a - \frac{g^2}{8}(\phi^a(x)\tau^a)^2 + O(g^3) \quad (3.6)$$

We shall use the following normalisation of the generators of $SU(N)$, (e.g. with constants defined for $SU(3)$ in [8]),

$$[\tau^a, \tau^b] = 2if^{abc}\tau^c \quad (3.7)$$

$$\frac{1}{2}\text{tr}[\tau^a\tau^b] = \delta^{ab} \quad (3.8)$$

We find,

$$S_H^{ad}(x) = \frac{1}{2}\text{tr}[\tau^a, \tau^d] - \frac{ig}{4}\phi^b(x)\text{tr}[\tau^a[\tau^b, \tau^d]] \quad (3.9)$$

$$\begin{aligned}
& -\frac{g^2}{16}\phi^b(x)\phi^c(x)\text{tr}[\tau^a[\tau^b, [\tau^c, \tau^d]]] + O(g^3) \\
& = \delta^{ad} - g f^{adb}\phi^b(x) - \frac{g^2}{2}f^{abe}f^{dce}\phi^b(x)\phi^c(x) + O(g^3)
\end{aligned}$$

Therefore, using (3.3) and considering the low momentum group elements, $U_L(x)$, to be slowly varying, such that $U_L^\dagger(x)U_L(y) \simeq 1$ and $S_L^{db}(x)S_L^{eb}(y) \simeq \delta^{de}$, we find,

$$\begin{aligned}
S^{ab}(x)S^{Tbc}(y) & \simeq S_H^{ad}(x)S_H^{cd}(y) \\
& = \delta^{ac} - g\Delta_1^{ac}(x, y) - \frac{g^2}{2}\Delta_2^{ac}(x, y) + O(g^3)
\end{aligned} \tag{3.10}$$

where

$$\Delta_1^{ac}(x, y) = f^{acb}(\phi^b(x) - \phi^b(y)) \tag{3.11}$$

$$\Delta_2^{ac}(x, y) = f^{abe}f^{cge}(\phi^b(x)\phi^g(x) + \phi^b(y)\phi^g(y) - 2\phi^b(x)\phi^g(y)) \tag{3.12}$$

We now re-write $\Delta^{ac}(x, y)$ as,

$$\begin{aligned}
\Delta^{ac}(x, y) & = G^{-1}(x - y)[2\delta^{ac}]^{-1}[1 - g\frac{[2\delta^{ac}]^{-1}}{\text{tr}[\delta^{aa}]} \Delta_1^{ac}(x, y) \\
& \quad - \frac{g^2}{2}\frac{[2\delta^{ac}]^{-1}}{\text{tr}[\delta^{aa}]} \Delta_2^{ac}(x, y) + O(g^3)]^{-1} \\
& = \frac{\delta^{ac}}{2}G^{-1}(x - y)[1 + \frac{g^2}{4\text{tr}[\delta^{aa}]} \Delta_2^{aa}(x, y)] + O(g^3)
\end{aligned} \tag{3.13}$$

where $[\delta^{ac}]^{-1} = \delta^{ac}$, $\delta^{ac}[\delta^{ac}]^{-1} = \text{tr}[\delta^{aa}]$. Also,

$$\Delta_1^{aa}(x, y) = 0 \tag{3.14}$$

$$\Delta_2^{aa}(x, y) = C_2(G)(\phi^b(x)\phi^b(x) + \phi^b(y)\phi^b(y) - 2\phi^b(x)\phi^b(y)) \tag{3.15}$$

$C_2(G)$ is the second Casimir operator and G denotes the adjoint representation in this case. For $\text{SU}(N)$, $C_2(G) = N$. Equation (3.15) is obtained using the relation,

$$f^{acd}f^{bcd} = C_2(G)\delta^{ab} \tag{3.16}$$

Next we need to evaluate $\lambda_i^a(x)$ up to $O(g^2)$. Using (3.4), (3.9) and following similar intermediary steps to the calculation of (3.9), we find,

$$\lambda_{i,H}^a(x) = -\partial_i \phi^a(x) + O(g^3) \quad (3.17)$$

$$\begin{aligned} \lambda_i^a(x) &= \lambda_{i,L}^a(x) - \partial_i \phi^a(x) - g f^{abc} \lambda_{i,L}^b(x) \phi^c(x) \\ &\quad - \frac{g^2}{2} f^{ace} f^{bde} \lambda_{i,L}^b(x) \phi^c(x) \phi^d(x) + O(g^3) \end{aligned} \quad (3.18)$$

Therefore, we can now write the effective action as,

$$\Gamma[U] = \int d^3x d^3y \Gamma[x, y] \quad (3.19)$$

where

$$\begin{aligned} \Gamma(x, y) &= \frac{1}{4} \partial_i \phi^a(x) G^{-1}(x - y) \partial_i \phi^a(y) \\ &\quad + \frac{1}{4} \lambda_{i,L}^a(x) G^{-1}(x - y) \lambda_{i,L}^a(y) \\ &\quad + \frac{g}{4} f^{abg} [\lambda_{i,L}^b(x) \phi^g(x) G^{-1}(x - y) \partial_i \phi^a(y) \\ &\quad + \partial_i \phi^a(x) G^{-1}(x - y) \lambda_{i,L}^b(y) \phi^g(y)] \\ &\quad - \frac{g^2}{8} f^{age} f^{bde} \lambda_{i,L}^b(x) G^{-1}(x - y) \lambda_{i,L}^a(y) [\phi^g(x) \phi^d(x) \\ &\quad + \phi^g(y) \phi^d(y) - 2\phi^g(x) \phi^d(y)] \\ &\quad + \frac{g^2}{16} \lambda_{i,L}^a(x) G^{-1}(x - y) \lambda_{i,L}^a(y) \frac{C_2(G)}{\text{tr}[\delta^{aa}]} [\phi^b(x) \phi^b(x) \\ &\quad + \phi^b(y) \phi^b(y) - 2\phi^b(x) \phi^b(y)] \end{aligned} \quad (3.20)$$

where $\Gamma(x, y)$ is the effective Lagrangian.

3.2 Renormalisation group transformation.

The renormalisation group transformation is performed by integrating over the high momentum dependent field, $\phi^a(x)$. This is akin to integrating over the fluctuating (quantum) fields in the background field method.

We consider

$$\Gamma^0(x, y) = \frac{1}{4} \partial_i \phi^a(x) G^{-1}(x - y) \partial_i \phi^a(y) \quad (3.21)$$

to be the zeroth order Lagrangian for the field $\phi^a(x)$. All other terms involving ϕ^a are treated as perturbations of this Lagrangian. (3.21) implies the two-point Green's function

$$\langle \phi^a(x) \phi^b(y) \rangle = 2\delta^{ab} [\partial_i^x \partial_i^y G^{-1}(x - y)]^{-1} \quad (3.22)$$

$G^{-1}(x - y)$ is defined in (2.14). So, because $\phi^a(x)$ is defined to have Fourier components $k > M$, we can write,

$$\langle \phi^a(x) \phi^b(y) \rangle = 2\delta^{ab} \int \frac{d^3 k}{(2\pi)^3} \frac{\exp[ik \cdot (x - y)]}{k^3} \quad (3.23)$$

with the integral performed over the limits $M < k < \Lambda$, $0 < \phi < 2\pi$, $0 < \theta < \pi$, where Λ is the ultra-violet cut-off.

Treating terms other than Γ^0 as perturbations, we see that

$$\int D\phi \exp\left[\int d^3 x d^3 y \{\Gamma^0(x, y) + F[\phi]\}\right] = \exp\left[\int d^3 x d^3 y \langle F[\phi] \rangle\right] \quad (3.24)$$

So we now write the effective Lagrangian as

$$\begin{aligned} \Gamma_L(x, y) = & \frac{1}{4} \lambda_{i,L}^a(x) G^{-1}(x - y) \lambda_{i,L}^a(y) \\ & + \frac{g}{4} f^{abg} [\lambda_{i,L}^b(x) G^{-1}(x - y) \langle \phi^g(x) \partial_i \phi^a(y) \rangle \\ & + \langle \partial_i \phi^a(x) \phi^g(y) \rangle G^{-1}(x - y) \lambda_{i,L}^b(y)] \\ & - \frac{g^2}{8} f^{age} f^{bde} \lambda_{i,L}^b(x) G^{-1}(x - y) \lambda_{i,L}^a(y) [\langle \phi^g(x) \phi^d(x) \rangle \\ & + \langle \phi^g(y) \phi^d(y) \rangle - 2 \langle \phi^g(x) \phi^d(y) \rangle] \\ & + \frac{g^2}{16} \lambda_{i,L}^a(x) G^{-1}(x - y) \lambda_{i,L}^a(y) \frac{C_2(G)}{\text{tr}[\delta^{aa}]} [\langle \phi^b(x) \phi^b(x) \rangle \\ & + \langle \phi^b(y) \phi^b(y) \rangle - 2 \langle \phi^b(x) \phi^b(y) \rangle] \end{aligned} \quad (3.25)$$

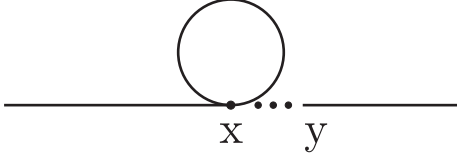


Fig. 1

First considering the terms of $O(g)$ we see that both their contributions are zero. This is explicitly seen as both of the correlations $\langle \phi^g(x) \partial_i \phi^a(y) \rangle$ and $\langle \partial_i \phi^a(x) \phi^g(y) \rangle$ have the structure δ^{ag} preceded by the totally antisymmetric group structure constant f^{abg} .

Considering the terms of $O(g^2)$, we need to examine the evaluation of (3.23) in some detail. First, we note that,

$$\langle \phi^a(x) \phi^b(x) \rangle = \langle \phi^a(y) \phi^b(y) \rangle = \frac{\delta^{ab}}{\pi^2} \log \frac{\Lambda}{M} \quad (3.26)$$

which occurs in (3.26) in terms such as

$$\frac{g^2}{16} \lambda_{i,L}^a(x) G^{-1}(x-y) \lambda_{i,L}^a(y) \frac{C_2(G)}{\text{tr}[\delta^{aa}]} \langle \phi^b(x) \phi^b(x) \rangle \quad (3.27)$$

This can be represented by a Feynman diagram, Fig 1, which shows a tadpole diagram. The external lines correspond to low momentum fields, $U(x)$, and the internal loop represents the integration over high momentum fields. The dotted line corresponds to $G^{-1}(x-y)$. It is important to note that in the region corresponding to $k < M$, $G^{-1}(x-y) = M\delta(x-y)$ and the propagator becomes local, associating either end of the dotted line, and the standard tadpole diagram is recovered. The evaluation of (3.26) is, however, the same at all scales of spatial separation, $|x-y|$.

To evaluate the other terms in $\Gamma_L(x, y)$, we write (3.23) in the form,

$$\langle \phi^a(x) \phi^b(y) \rangle = \frac{\delta^{ab}}{\pi^2} \int_{M|x-y|}^{\Lambda|x-y|} dt \frac{\sin t}{t^2} \quad (3.28)$$

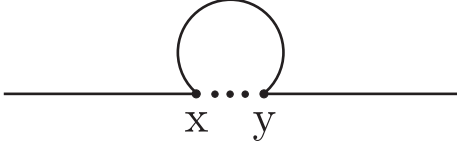


Fig. 2

where the change of variable $t = k|x - y|$ has been made. t is a dimensionless variable.

We shall introduce a scale, μ , such that for $t < \mu$, $\sin t \simeq t$. This allows us to write,

$$\langle \phi^a(x) \phi^b(y) \rangle = \begin{cases} a & M|x - y| > \mu \\ -\frac{\delta^{ab}}{\pi^2} \log[M|x - y|] + b & M|x - y| < \mu \end{cases} \quad (3.29)$$

where a and b are finite contributions which are independent of M and Λ and hence are ignored in the following. All μ dependence is in the finite terms.

The correlations $\langle \phi^a(x) \phi^b(y) \rangle$ appear in $\Gamma_L(x, y)$ in terms such as,

$$-\frac{g^2}{8} \lambda_{i,L}^a(x) G^{-1}(x - y) \lambda_{i,L}^a(y) \frac{C_2(G)}{\text{tr}[\delta^{aa}]} \langle \phi^b(x) \phi^b(y) \rangle \quad (3.30)$$

which can also be interpreted in terms of Feynman diagrams. Fig. 2 shows the diagram corresponding to (3.30). It depicts a horse-shoe. As in Fig. 1, the external lines correspond to the low momentum fields, $U_L(x)$, the horse-shoe represents the integration over the high momentum fields and the dotted line again represents $G^{-1}(x - y)$. The horse-shoe diagram is unimportant in the region $\frac{\mu}{M} < |x - y|$ as it only gives a finite contribution, independent of a cut-off or M . In the region $0 < |x - y| < \frac{\mu}{M}$ the horse-shoe diagram is important and contributes $\frac{\delta^{ab}}{\pi^2} \log \frac{1}{M|x - y|}$, (3.29). We must not here immediately interpret $|x - y|$ to therefore be the inverse of the cut-off but rather take our interpretation from the form of the resulting Lagrangian.

Substitution of (3.26) and (3.29) into $\Gamma_L(x, y)$ yields the effective action for the low momentum modes with a renormalised coupling constant, \tilde{g} . With $\lambda_{i,L}^a = \frac{i}{\tilde{g}} \text{tr}[\tau^a U_L^+ \partial_i U_L]$

this can now be written as,

$$\begin{aligned}\Gamma_L(x, y) &= \frac{1}{4}\lambda_{i,L}^a(x)G^{-1}(x-y)\lambda_{i,L}^a(y) \\ &= \frac{1}{2\tilde{g}^2}\text{tr}[\partial_i U_L(x)G^{-1}(x-y)\partial_i U_L^\dagger(y)]\end{aligned}\tag{3.31}$$

where

$$\tilde{g}^2 = \begin{cases} g^2(1 + \frac{g^2}{2\pi^2}C_2(G)\log\frac{\Lambda}{M}) & M|x-y| > \mu \\ g^2(1 + \frac{g^2}{2\pi^2}C_2(G)\log\Lambda|x-y|) & M|x-y| < \mu \end{cases}\tag{3.32}$$

$\beta(g)$ is calculated using the standard definition,

$$\beta(g) = M\frac{\partial}{\partial M}\tilde{g}|_{g,\Lambda}\tag{3.33}$$

but first we need to interpret the appearance of $|x-y|$ in (3.32). For $|x-y| > \frac{\mu}{M}$, M is the renormalisation scale, whereas, for $|x-y| < \frac{\mu}{M}$, $\frac{1}{|x-y|}$ should be interpreted as the renormalisation scale. Therefore, we can re-write \tilde{g} as,

$$\tilde{g} = g + \frac{g^3}{(4\pi)^2}4C_2(G)\log\frac{\Lambda}{M'} + O(g^5)\tag{3.34}$$

where

$$M' = \begin{cases} M & |x-y| > \frac{\mu}{M} \\ \frac{1}{|x-y|} & |x-y| < \frac{\mu}{M} \end{cases}\tag{3.35}$$

Thus we obtain,

$$\beta(g) = M'\frac{\partial}{\partial M'}\tilde{g}|_{g,\Lambda} = -\frac{g^3}{(4\pi)^2}4C_2(G) + O(g^5)\tag{3.36}$$

This should be compared to the standard β -function for QCD, e.g. [9],

$$\beta(g) = -\frac{g^3}{(4\pi)^2}[(4 - \frac{1}{3})C_2(G) - \frac{2}{3}n_f] + O(g^5)\tag{3.37}$$

In this paper and in [3], a variational approach to QCD has been considered in the absence of fermions, $n_f = 0$. With a gauge group $SU(N)$, $C_2(G) = N$. The discrepancy between 4

in the result of this paper and the $\frac{11}{3}$ of the standard QCD β -function quoted above is due to the fact that only longitudinal gluons are included in the variational ansatz and that transverse gluons, which would contribute the extra $\frac{1}{3}$, are omitted. This is as expected because in the creation of the ansatz it was required that the wavefunctional obey Gauss' law, which is indeed satisfied by longitudinal photons with a source.

The representation of a β -function coefficient as the sum of two contributions proportional to -4 and $1/3$ is not new at all. It has been known for a long time, [10] and [11] that in the background field method the contribution of charged particles with spin S to the one-loop β -function is given by

$$\beta_S(g) = -g^3 \frac{(-1)^{2S}}{(4\pi)^2} [(2S)^2 - \frac{1}{3}] \quad (3.38)$$

where we omit the group factor. The asymptotically free (for integer spins) “spin” factor $4S^2$, where $S = 1$ for the vector field A , gives 4 which is precisely what we have obtained in our calculations. As was discussed in [11], this spin factor is related to the influence of the background field on the electric dipole moment density, which includes contributions from the time-like polarization states of the massless vector fields in the case of the Feynman gauge. In a Coulomb gauge, where the time-like modes do not exist as dynamical degrees of freedom, one can see that the electric dipole density appears because of Gauss' law. This explains why in our method, where the Gauss' law is implemented by construction, we obtained the same result.

It is interesting to note that the same decomposition $4 - 1/3$ comes from the results for the calculation of the pre-exponential factor, or the renormalisation of the charge, for the BPST instanton within the path integral formalism of QCD, [12]. In this work, the vector field is split into components, $A_\mu^a = A_\mu^{a(inst)} + a_\mu^a$, and the action expanded in terms of the deviation a_μ^a from the instanton field $A_\mu^{a(inst)}$. Analysis of the resulting path integral

and examination of the contribution of the zero-frequency modes yields,

$$\frac{\langle 0|0_T \rangle_{ins}^{Reg}}{\langle 0|0_T \rangle_{p.th.}} = const. \int \frac{d^4 x d\rho}{\rho^5} S_0^4 \exp[-S_0 + 8 \log M\rho + \Phi_1] \quad (3.39)$$

where p. th. refers to perturbation theory, $|0 \rangle_T$ is the vacuum after time T, $d^4 x$ is the measure of integration over the four coordinates of the centre of the instanton, ρ is the scale of the instanton, M is the introduced cut-off parameter and $S_0 = \frac{8\pi^2}{g^2}$. Φ_1 denotes the contribution of the positive frequency modes. In the limit $M\rho \gg 1$, Φ_1 was evaluated to the one loop level by means of ordinary perturbation theory to be

$$\Phi_1 = \frac{2}{3} \log M\rho \quad (3.40)$$

Substituting (3.40) into the argument of the exponential in (3.39), the result of the renormalised charge is obtained,

$$\frac{8\pi^2}{g^2(\rho)} = \frac{8\pi^2}{g_0^2} - 2\left(4 - \frac{1}{3}\right) \log M\rho \quad (3.41)$$

The 4 explicitly came from the evaluation of the zero frequency modes and the $-\frac{1}{3}$ from the evaluation of the transverse positive frequency modes.

4. Conclusion

The variational ansatz of [3] has been studied and the renormalisation of the effective charge in the wavefunctional has been carried out up to $O(g^2)$. In this procedure, the effective action is a non-local, non-linear sigma model in three dimensions where the fields considered are the group elements of the original gauge transformation. The group elements are decomposed into low and high momentum dependent components and the

renormalisation transformation is effected by integrating over the high momentum dependent modes up to (quadratic) terms of $O(g^2)$. The β function is found to be

$$\beta(g) = -\frac{g^3}{(4\pi)^2} 4C_2(G) + O(g^5) \quad (4.1)$$

This should be directly compared with the known β function for QCD, [9];

$$\beta(g) = -\frac{g^3}{(4\pi)^2} [(4 - \frac{1}{3})C_2(G) - \frac{2}{3}n_f C(r)] + O(g^5) \quad (4.2)$$

The only difference between the two (in the absence of fermions $n_f = 0$) is the inclusion of the factor $\frac{1}{3}$ in the latter, which is due to the screening effect of virtual transverse gluons. Only longitudinal gluons were incorporated in the ansatz as they satisfy Gauss' law, the constraint ensuring gauge invariance of the wavefunctional.

The interpretation of the renormalisation procedure by considering the Feynman diagrams shows that a vital contribution is made by tadpole diagrams and new 'horse-shoe' diagrams which must be interpreted with the non-local propagator $G^{-1}(x - y)$. The renormalisation scale is found to be M when $|x - y| > \frac{1}{M}$ and $\frac{1}{|x-y|}$ when $|x - y| < \frac{1}{M}$, where M is a dynamically generated mass scale (and we have taken μ , a dimensionless constant from the previous section, to be 1 for simplicity). The explicit calculations made in this paper give a proof of the conjecture made in [3] that to calculate the gap in a variational approach one can use an effective QCD coupling constant and that this gap will automatically be related to Λ_{QCD} .

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